The Code Equivalence Problem: New Algorithms and Reductions

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Based on joint works with Drisana Bhatia, Jean-François Biasse, Medha Durisheti, Lucas LaBuff, Kaung Myat Htay Win, Vincenzo Pallozzi Lavorante, and Philip Waitkevich.

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This talk is based on two joint works

1. **Asymptotic improvements to provable algorithms for the code equivalence problem**
   with Drisana Bhatia, Jean-François Biasse, Medha Durisheti,
   Lucas LaBuff, Vincenzo Pallozzi Lavorante, and Philip Waitkevich
   (https://eprint.iacr.org/2025/187).
   In ISIT 2025 and accepted to IEEE Transactions on Information Theory.

2. **Relating Code Equivalence to Other Isomorphism Problems**
   with Kaung Myat Htay Win (https://eprint.iacr.org/2024/782).
   In Designs, Codes, and Cryptography 2025.

Computational Isomorphism Problems

Problem: Given two {graphs, codes, lattices} as input, decide if they are "essentially the same."

Graphs

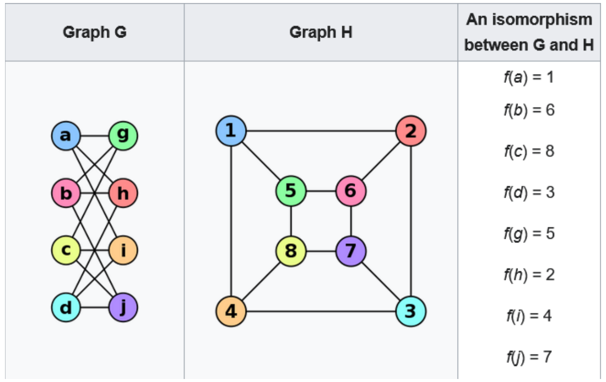
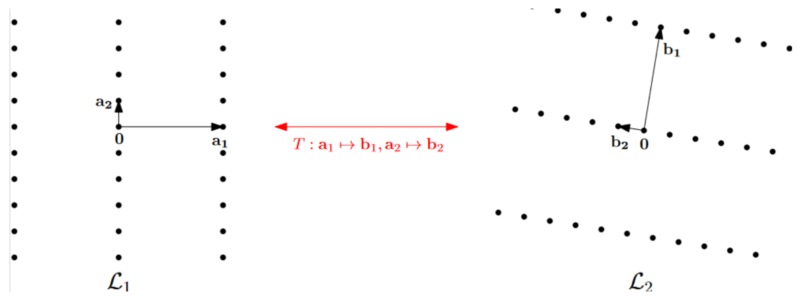


Image source: Wikipedia.

Codes

$$G = \begin{pmatrix} -1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} \mapsto G' = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & -1 \end{pmatrix}$$

Lattices



Goals of our work:

1. Give faster algorithms for code equivalence.

2. Understand the relationship between code equivalence.
and isomorphism problems on graphs and lattices.

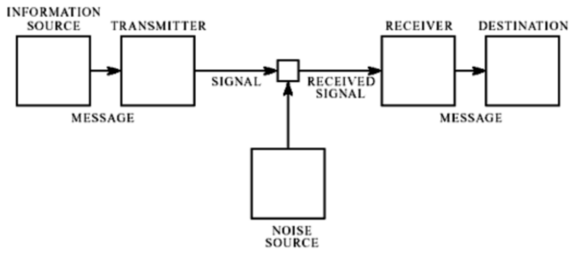
Cryptographic Motivation

Cryptography Based on Code Equivalence:

- McEliece Cryptosystem [McEliece, ’78].
- “Classic McEliece” NIST PQC Standardization Process Submission [Albrecht et al., ’22].
- LESS Identification Scheme [Biasse, Micheli, Persichetti, Santini, ’20].

Cryptography Based on Lattice Isomorphism:

- LIP-based KEM: (Ducas and van Woerden, ‘22).
- Rotations of $\mathbb{Z}^n$ PKC: [**Bennett**, Ganju, Peetathawatchai, and Stephens-Davidowitz, ’23].
- HAWK Digital Signature Scheme: [Ducas, Postlethwaite, Pulles, van Woerden, ’22].



From “A Mathematical Theory of Communication”

Coding Theory 101

Main use of error-correcting codes: Robust communication.

Want to encode a k-bit message in a redundant way.

Ex. 4-bit message m. Repeat each coordinate 3 times.
- $m \coloneqq (0,1,0,0) \mapsto c \coloneqq (0,0,0,1,1,1,0,0,0,0,0,0)$.
- c is $n = 12$ bits long, protects against 1 arbitrary error.

Ex. Hamming(7,4) code. Compute $C = Gm$, where $G \coloneqq \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.
- $m \coloneqq (0,1,0,0) \mapsto c \coloneqq (1,0,0,1,1,0,0)$.
- c is $n = 7$ bits long, protects against 1 arbitrary error.

Want a code with n as *small* as possible and # of errors tolerated as *large* as possible.

Codes

Def. An $[n, k, d]_q$ code C is a linear subspace of $\mathbb{F}_q^n$ of dimension k with $\|x - y\|_0 \geq d$ for distinct $x, y \in C$.

- $\|\cdot\|_0$ denotes Hamming weight, the number of non-zero coordinates of a vector.
- $[n, k]_q$ denotes an $[n, k, d]_q$ code for some d.

Primal representation: Column basis generator matrix $G \in \mathbb{F}_q^{n \times k}$, $C(G) \coloneqq \left\{G\boldsymbol{z} : \boldsymbol{z} \in \mathbb{F}_q^k\right\}$.

Fact: $C(G_1) = C(G_2)$ if and only if $G_2 = G_1 U$ for an invertible matrix U (i.e., $U \in \mathrm{GL}_k(\mathbb{F}_q)$).

Code Equivalence Problem(s)

C_1, C_2 are *linearly equivalent* if there exists a monomial matrix M such that $MC_1 = C_2$.

- A monomial matrix M is such that $M = DP$ for full-rank diagonal D and permutation matrix P.

Ex. $$\begin{pmatrix} 0 & 0 & 3 \\ 1 & 0 & 0 \\ 0 & 4 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

C_1, C_2 are *permutationally equivalent* if there exists a permutation matrix P such that $PC_1 = C_2$.

Permutationally Equiv.:

$$G = \begin{pmatrix} 2 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}, G' = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 2 \end{pmatrix}$$

Linearly Perm., Not Perm. Equiv.:

$$G = \begin{pmatrix} 2 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, G' = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Not Linearly Equiv.:

$$G = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}, G' = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Search Versions of Code Equivalence

Def. Linear Code Equivalence Problem (LCE) over $\mathbb{F}_q$: Given generator matrices G_1, G_2 of equivalent $[n, k]_q$ codes, find an $n \times n$ monomial matrix M and some $U \in \mathrm{GL}_k(\mathbb{F}_q)$ such that $MG_1U = G_2$ (if they exist).

Def. Permutation Code Equivalence Problem (PCE) over $\mathbb{F}_q$: Given generator matrices G_1, G_2 of equivalent $[n, k]_q$ codes, find an $n \times n$ permutation matrix P and some $U \in \mathrm{GL}_k(\mathbb{F}_q)$ such that $PG_1U = G_2$ (if they exist).

Def. Signed Permutation Code Equivalence (SPCE) over $\mathbb{F}_q$: Given generator matrices G_1, G_2 of $[n, k]$ codes, find an $n \times n$ signed permutation matrix S and some $U \in \mathrm{GL}_k\left(\mathbb{F}_q\right)$ such that $SG_1U = G_2$ (if they exist).

Graph Isomorphism Problem

Def. Graphs $G_1 = (V, E_1), G_2 = (V, E_2)$ are isomorphic if there exists a permutation $\pi : V \to V$ such that for all $u, v \in V$, $\{u, v\} \in E_1 \Leftrightarrow \{\pi(u), \pi(v)\} \in E_2$.

Graph Isomorphism Problem (GI): Decide if input graphs G_1, G_2 (represented by adjacency matrices) are isomorphic.

[Babai '16]: GI is solvable in $n^{\mathrm{poly}(\log n)}$ time.

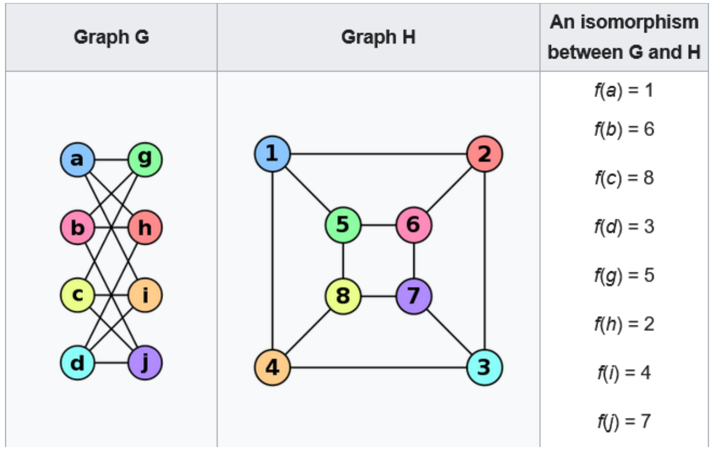


Image source: Wikipedia.

Part 1: Algorithms for Code Equivalence

Related Work

1. Heuristic algorithms using Information Set Decoding (ISD):
   a. [Leon ’82], [Beullens ’20], [Barenghi, Biasse, Persichetti and Santini ’23].
2. Algorithms for Codes with small hulls (the *hull* of C is $C \cap C^{\perp}$):
   a. Supporting splitting algorithm [Sendrier ’00].
   b. Reduction from [Bardet, Otmani and Saeed-Taha ’09].
3. Reduction from LCE on $[n, k]_q$ codes to PCE on $[(q-1)n, k]_q$ codes [Sendrier-Simos ’13].
4. **Provable general algorithms (our focus):**
   a. Deterministic PCE algorithm running in $2^{n+o(n)}$ time [Babai ‘11].
   b. Randomized LCE algorithm running in $2^{n/2+o(n)}$ time on *random* codes over fields of order $q \geq 7$ [Nowakowski ’25].

Question: What about *worst-case* codes and when $q < 7$?

Our Results

Theorem: The following algorithms exist for code equivalence over arbitrary $[n, k]_q$ codes for arbitrary prime powers q:

1. A $2^{n+o(n+q)}$-time deterministic algorithm for LCE.
2. A $2^{n/2+o(n+q)}$-time randomized algorithm for PCE and LCE.
3. A $2^{n/3+o(n+q)}$-time quantum algorithm for PCE and LCE.

Algorithm (1) complements the algorithm of [Babai ’11] for PCE.

Algorithms (2) and (3) resolve the question on the previous slide and remove both restrictions from algorithms with similar running times in [Nowakowski ’25].

High-level idea for (2) and (3): Reduce code equivalence to collision/claw finding.

Babai's PCE Algorithm

Let C be an $[n,k]_q$ code and let $G \in \mathbb{F}_q^{n\times k}$ be a generator matrix of C.

Def. An *information set* C is a set $S = \{i_1, \dots, i_k\} \subseteq [n]$ of coordinates such that the $k \times k$ matrix obtained by restricting G to rows indexed by S is full-rank.

Def. If $G = \begin{pmatrix} I_k \\ G' \end{pmatrix}$ for $G' \in \mathbb{F}_q^{(n-k)\times k}$ then it is said to be in *systematic form.*

- If $[k] \coloneqq \{1, \dots, k\}$ is an information set, then it is easy to put $G = \begin{pmatrix} G' \\ G'' \end{pmatrix}$ into systematic form by setting

$$G \coloneqq \begin{pmatrix} G' \\ G'' \end{pmatrix} \cdot (G')^{-1} = \begin{pmatrix} I_k \\ G'' \cdot (G')^{-1} \end{pmatrix}.$$

Def. Information sets J_1, J_2 of equivalent codes C_1, C_2 respectively are called *matching* if there exists a permutation $\pi: [n] \to [n]$ such that $\pi(C_1) = C_2$ and $\pi(J_1) = J_2$.

- **Observation:** A permutation such that $\pi(C_1) = C_2$ must map information sets to information sets.

Main idea of algorithm: To solve PCE, it suffices to know a pair of matching information sets and to make one call to a graph isomorphism (GI) oracle.

Babai's PCE Algorithm

Let G_1, G_2 be generator matrices of equivalent codes C_1, C_2 with matching information sets J_1, J_2.

Assume WLOG that $J_1 = J_2 = \{1, \ldots, k\}$. If not, permute coordinates of C_1, C_2 so that this holds.

Put G_1, G_2 into systematic form so that $G_1 = \begin{pmatrix} I_k \\ G_1' \end{pmatrix}, G_2 = \begin{pmatrix} I_k \\ G_2' \end{pmatrix}$.

Then by the assumption that J_1, J_2 are matching, there must exist $P_1 \in \mathcal{P}_k, P_2 \in \mathcal{P}_{n-k}$,
$U \in \mathrm{GL}_k(\mathbb{F}_q)$ such that

$$\begin{pmatrix} P_1^{-1} & 0 \\ 0 & P_2 \end{pmatrix} \begin{pmatrix} I_k \\ G_1' \end{pmatrix} U = \begin{pmatrix} I_k \\ G_2' \end{pmatrix}.$$

This implies that $U = P_1$ and therefore $P_2 G_1' P_1 = G_2'$.

Babai's PCE Algorithm

To recover a permutation from C_1 to C_2, it suffices to find permutation matrices P_1, P_2 such that $P_2 G_1' P_1 = G_2'$.

Observation [Babai 2011]: This is equivalent to graph isomorphism on $(n-k, k)$-bipartite graphs with $\mathbb{F}_q$-labeled edges!

- Regard G_1', G_2' as adjacency matrices of such graphs.

Theorem [Babai 2016]: There is a quasipolynomial-time algorithm for GI.

Babai's PCE algorithm:

- Compute an arbitrary information set J_1 of C_1.
- Enumerate all $\binom{n}{k} \leq 2^n$ size-k subsets of indices J_2 corresponding to candidate information sets of C_2 matching J_1.
- Solve the resulting GI instance in $2^{o(n)}$ time.

Takes $2^{n+o(n)}$ time.

Our Randomized $2^{n/2}$-time Algorithm for PCE

Theorem [Babai 2019]: There is a quasipolynomial-time computable *canonical form* for graphs.

- A canonical form $F: \mathbb{F}_q^{m\times n} \to \mathbb{F}_q^{m\times n}$ for $\mathbb{F}_q$-edge labeled bipartite graphs is a function such that:
  (1) $F(A) = A$,
  (2) $F(A_1) = F(A_2)$ if and only if A_1 and A_2 are adjacency matrices of isomorphic graphs.

Let f_i for $i = 1, 2$ be a function mapping information sets of C_i to $F\left(G_i'\right)$, where $G_i = \begin{pmatrix} I_k \\ G_i' \end{pmatrix}$.

- Interpret G_i' as the adjacency matrix of a bipartite graph.
- Note that f_1, f_2 have the same range.

Key idea: We have reduced PCE to finding a pair (J_1, J_2) of information sets of C_1, C_2 such that $f_1(J_1) = f_2(J_2)$.

- Such a pair (J_1, J_2) is called a *claw.*

Each code C_i has the same number $N \leq \binom{n}{k} \leq 2^n$ of information sets.

Our Randomized $2^{n/2}$-time Algorithm for PCE

Sample m independent, uniformly random information sets from each of C_1, C_2.

- I.e., sample information sets $J_1, \ldots, J_m$ of C_1 and $J'_1, \ldots, J'_m$ of C_2.
- The expected number of matching information sets/claws (J_i, J'_j) is at least m^2/N.

So, setting $m \approx N^{1/2}$, we get that the expected number of claws is at least 1.

- Runs in roughly $N^{1/2} \leq 2^{n/2}$ time.

Issue #1: Showing that this works with high probability.

Solution: Bound variance of expected number of claws, apply Chebyshev's inequality to show that it concentrates around expectation.

Issue #2: How do we sample uniformly random information sets?

Solution: We don't. Instead, we use an algorithm for matroid basis sampling [Anari, Liu, Oveis Gharan, Vinzant '19] that efficiently samples *nearly* uniformly random bases efficiently.

Extension: We also extend this to a $2^{n/3}$-time quantum algorithm for PCE.

Thank you!